Two-Warehouse Inventory Model with Stock-Dependent Demand for Deteriorating Items with Shortages

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Previous literature on two-warehouse inventory model has always assumed that inventory holding cost in Rented Warehouse (RW) is greater than Owned Warehouse (OW). This resulted in a Last-In-First-Out (LIFO) flow of inventory that items in RW must be consumed prior to OW to avoid higher holding cost. In this paper, a two-warehouse inventory model for perishable inventory items with the First-In-First-Out (FIFO) dispatching policy is developed. Pakkala and Achary’s two-warehouse LIFO inventory model is first modified and then a FIFO dispatching two-warehouse inventory model with deterioration is proposed. The deterioration rate in OW is time dependent and in RW is constant, and the time-dependent demand rate is applied in both warehouses. On comparison the two inventory models indicated that the FIFO model is less expensive to operate than the LIFO model, if the mixed effect of deterioration and holding cost in RW is less than that of OW. The paper also finds the particular case when the deterioration rate in both warehouses is constant, which is different and supports the theorem in both cases.

Keywords: Inventory, Two-warehouse, Deterioration, Stock-dependent demand, FIFO, LIFO

Introduction

Economic Order Quantity (EOQ) model is formulated by considering three inventory costs to achieve a minimum cost system. These costs are the procurement cost, carrying cost and shortage cost. One of the unrealistic assumptions is that items are not perishable while in shortage. However, there are items, such as highly volatile substances, radioactive materials and fresh goods, in which the rate of deterioration is higher. Loss from deterioration should not be ignored. Ghare and Schrader (1963) were the first to consider issues regarding ongoing deterioration of inventory. Since then, research for deterioration of inventory has extensively been done by many

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researchers from time to time. Raafat (1991) and Goyal and Giri (2001) have made excellent reviews of these models.

An interesting research topic incorporating deterioration effect in inventory decision involves the situations in which there are two storage facilities. Sarma (1987) is the first to discuss the two-warehouse inventory model with deterioration. In his model, a single inventory item is first stored in the Owned Warehouse (OW), with limited capacity, and any additional quantity to be stored in the Rented Warehouse (RW). An infinite replenishment rate is considered in this model with uniform scheduling period and shortage allowance. Other authors, e.g., Goswami and Chaudhuri (1992), Benkherouf (1997), Bhunia and Maiti (1998), and Lee and Ma (2000) proposed the two-warehouse models when demand is a function of time either with or without the consideration of deterioration. Pakkala and Achary (1992) extended Sarma’s model to the case of finite replenishment rate with shortage. All the above-mentioned research models are commonly referred to as continue release model, assuming that inventory is to be released directly and continuously in each warehouse. Murdeshwar and Sathe (1985) and Pakkala and Achary (1994) considered bulk release model in which inventory in RW must first be transferred to OW before its release to the customer. Bhunia and Maiti (2001) suggested a two level inventory model for linear trend in demand and a fixed time horizon demand rate. Lang and Zhau (2003) proposed a multi-warehouse inventory model for items with time varying demand and shortages. Lang and Zhau (2005) developed a two-warehouse inventory model for items with stock level dependent.

It is generally assumed that RW offers better preserving facilities than OW, therefore it charges a higher holding cost. The two-warehouse models discussed above naturally adopt the Last-In-First-Out (LIFO) inventory flow. Under such circumstances, inventories are first stored in OW, with overflows going to RW. But when retrieving for consumption, it should always start from RW, when available, before retrieving from OW. First in RW, particular in a public warehouse, a professional vendor who specializes in the warehousing operation would carry a lower operating cost due to well-equipped setups, learning effect of trained worker and the economics of scale from higher volume. Second, as competition increases between warehouse facilities in real world, their ability to offer value added service with competitive lower price becomes more and more necessary. Many businesses have gotten into or expanded their use of public warehouse because of cheap shipping or other financial reasons. Finally, a critical point of inventory decision for perishable products: to allow later stored inventory in RW to be dispatched last means a greater risk of deterioration of inventory. The cost of deteriorated inventory and related opportunity cost may exceed the cost saving benefit derived from the warehouse rent. In the real world, maintaining a FIFO rule of inventory flow has been the common practice of most managers. In fact, Pierskalla and Roach (1972) have
shown that a FIFO issuing policy is optimal for perishable and deteriorating inventories in a single warehouse setting with unlimited capacity.

In this paper, Pakkala and Achary’s (1992) LIFO inventory model (Figure 1) for deteriorating items with two warehouses and finite replenishment rate will be reconsidered. LIFO model is first modified, and we propose a FIFO two-warehouse model in which the inventory in OW—which is stored first—will be consumed before that in RW, based on the above considerations that the holding cost in RW is not necessarily greater than in OW. The demand rate in both warehouses is time dependent and the deterioration in OW is time dependent and in RW is constant. Comparison of the two models indicated that the FIFO model is less expensive to operate than LIFO, if the mixed effect of deterioration and holding cost in RW is less than that of OW.

Figure 1: Inventory Level of Pakkala and Achary’s Two-Warehouse LIFO Model

2. Notations and Assumptions

To develop the mathematical model of inventory replenishment schedule with two warehouses, the notation adopted in this paper is as below:

\[
D(t) \quad \text{Demand rate which is taken as exponential increasing function of time, i.e., } D(t) = ae^{\lambda t} \text{ where } a \text{ is initial demand and } \lambda \geq 0
\]

\[
P(t) \quad \text{Production rate such that } P(t) = \gamma D(t), \text{ where } \gamma > 1.
\]

\[
\alpha, \beta \quad \text{Deterioration rates in OW and RW respectively } 0 < \alpha, \beta < 1.
\]

\[
W \quad \text{Capacity of OW}
\]
$R$ Maximum inventory level in RW

$B$ Maximum shortage allowed

$h_1, h_2$ Holding cost held in OW and RW respectively

$C_1$ Cost of deteriorated item

$C_2$ Shortage cost per unit per unit time

$C_3$ Opportunity cost (i.e., goodwill cost) per unit per unit time

$T_i, T'_i$ Time period in a production cycle of stage $i$, $i = 1, 2 ... 6$ for LIFO and FIFO models respectively

$C'$ Unit set up cost

$I_i(t), q_i(t)$ Inventory level at any time for LIFO and FIFO models respectively

$\delta$ Backlogging rate

In addition, the following assumptions are imposed:

• Production rate is dependent on the demand of the item, which is variable, and lead time is zero.

• The RW has unlimited capacity; the OW has a fixed capacity of units.

• Inventory items are stored in RW only after OW is fully utilized. Once stored, these items are assumed not to be relocated.

• Shortages are allowed. Unsatisfied demand is backlogged, and the fraction of shortage backordered is $\delta$, which is a positive constant and $0 < \delta < 1$.

3. LIFO Inventory Model

The inventory in a production system with LIFO dispatching policy is depicted in Figure 2. The inventory cycle can be divided into six parts, named $T_{Li}, i = 1, 2 ... 6$. Initially, $B_L$ units of backorders are carried over from the previous cycle. The production run starts at the beginning of $T_{L1}$, and while production and demand happen simultaneously, backorders are made up within $T_{L1}$ at the rate of $P - D$. During $T_{L2}$, inventory items in OW are built from 0 up to $W$ units with a deterioration rate of $\alpha t$. Any production quality exceeding this level must be stored in RW. During $T_{L3}$, inventory items in RW are built from 0 to $R_L$ units but with a deterioration rate of $\beta$. Meanwhile, in OW, inventory level will be depleted because of deterioration in stock with a rate of $\alpha t$. 

The production run stops at the end of $T_{L3}$, and $R_L$ units of inventory items in RW are depleted in $T_{L4}$. The remaining inventory items in OW are then depleted in $T_{L5}$ by demand and deterioration. Finally, $B_L$ units of backorders are accumulated at the end of $T_{L6}$ by a rate of $D$, which completes the production cycle. In this system, the management seeks to find the optimal levels of both $R_L$ and $B_L$.

The inventory levels at any time in the production cycle are governed by the following differential equations:

1. $I'_{L1}(t) = P(t) - D(t)$, $0 \leq t \leq T_{L1}$ ...

2. $I'_{L2}(t) + \alpha t I_{L2}(t) = P(t) - D(t)$, $0 \leq t \leq T_{L2}$ ...

3. $I'_{L3}(t) = 0$, $0 \leq t \leq T_{L3}$ ...

4. $I'_{L4}(t) + \beta I_{L4}(t) = P(t) - D(t) - \alpha w$, $0 \leq t \leq T_{L5}$ ...

5. $I'_{L5}(t) + \beta I_{L5}(t) = -D(t)$, $0 \leq t \leq T_{L4}$ ...

6. $I'_{L6}(t) + \alpha I_{L6}(t) = 0$, $0 \leq t \leq T_{L4}$ ...

7. $I'_{L7}(t) + \alpha t I_{L7}(t) = -D(t)$, $0 \leq t \leq T_{L5}$ ...

8. $I'_{L8}(t) = -\delta D(t)$, $0 \leq t \leq T_{L6}$ ...
Using the boundary conditions that $I_{L_1}(T_{L_1}) = 0$, $I_{L_2}(0) = 0$, $I_{L_4}(0) = 0$, $I_{L_4}(T_{L_4}) = 0$, $I_{L_6}(0) = W$, $I_{L_7}(T_{L_5}) = 0$, and $I_{L_8}(0) = 0$, the above equations can be solved respectively as follows:

$$I_{L_1}(t) = \frac{(\gamma - 1)a}{b} \left[ e^{bt} - e^{\beta t_{L_1}} \right], \quad 0 \leq t \leq T_{L_1} \quad \ldots(9)$$

$$I_{L_2}(t) = (\gamma - 1)a \left[ t + \frac{b}{2} t^2 + \frac{\alpha}{6} t^3 \right] e^{\frac{\alpha t^2}{2}}, \quad 0 \leq t \leq T_{L_2} \quad \ldots(10)$$

$$I_{L_3}(t) = W, \quad 0 \leq t \leq T_{L_3} \quad \ldots(11)$$

$$I_{L_4}(t) = \frac{(\gamma - 1)a}{b + \beta} \left( e^{bt} - e^{-\beta t} \right) - \frac{\alpha W}{\beta} (1 - e^{-\beta t}) \quad 0 \leq t \leq T_{L_3} \quad \ldots(12)$$

$$I_{L_5}(t) = \frac{a}{b + \beta} \left[ e^{(b + \beta) t_{L_4}} - e^{bt} \right], \quad 0 \leq t \leq T_{L_4} \quad \ldots(13)$$

$$L_{L_6}(t) = W e^{\frac{\alpha t^2}{2}}, \quad 0 \leq t \leq T_{L_4} \quad \ldots(14)$$

$$I_{L_7}(t) = a \left[ (T_{L_5} - t) + \frac{b}{2} T_{L_5}^2 - t^2 \right] + \frac{\alpha}{6} \left( T_{L_5}^3 - t^3 \right) e^{\frac{\alpha t^2}{2}}, \quad 0 \leq t \leq T_{L_5} \quad \ldots(15)$$

$$L_{L_8}(t) = \frac{\delta a}{b} (1 - e^{\beta t}), \quad 0 \leq t \leq T_{L_6} \quad \ldots(16)$$

Now, the inventory holding cost per cycle in OW and RW is:

$$H_1 = h_1 \left[ \int_0^{T_{L_2}} I_{L_2}(t) dt + \int_0^{T_{L_3}} I_{L_3}(t) dt + \int_0^{T_{L_4}} I_{L_4}(t) dt + \int_0^{T_{L_5}} I_{L_7}(t) dt \right]$$

$$= h_1 \left[ (\gamma - 1)a \left( \frac{T_{L_2}^2}{2} + \frac{b}{2} T_{L_2}^3 - \frac{\alpha}{6} T_{L_2}^4 - \frac{\alpha b}{20} T_{L_2}^5 \right) \right.$$

$$+ W (T_{L_3} + T_{L_4}) - \frac{\alpha W}{t} T_{L_4}^3$$

$$\left. + a \left( \frac{1}{2} T_{L_5}^2 + \frac{b}{3} T_{L_5}^3 + \frac{\alpha}{12} T_{L_5}^4 - \frac{\alpha b}{30} T_{L_5}^5 \right) \right] \quad \ldots(17)$$
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\[ H_2 = h_2 \left\{ \int_0^{T_{L3}} I_{L4}(t) dt + \int_0^{T_{L5}} I_{L5}(t) dt \right\} \]

\[ = h_2 \left[ \frac{a}{b \beta (b + \beta)} \left\{ (\gamma - 1)(\beta e^{\beta T_{L3}} + b e^{-\beta T_{L3}} - b - \beta) + b e^{(b + \beta) T_{L4}} \right\} \right. \]

\[ \left. - (b + \beta) e^{b T_{L4}} + \beta \right\} - \frac{\alpha W}{\beta^2} \left( e^{-\beta T_{L3}} + \beta T_{L3} - 1 \right) \]  \hspace{1cm} \text{(18)}

The cost of deteriorated items per cycle,

\[ C_D = C_1 \left[ \frac{\alpha t}{h_1} H_1 + \frac{\beta}{h_2} H_2 \right] \]  \hspace{1cm} \text{(19)}

The shortage cost per cycle,

\[ C_S = -C_2 \left\{ \int_0^{T_{L1}} I_{L1}(t) dt + \int_0^{T_{L5}} I_{L5}(t) dt \right\} \]

\[ = \frac{C_s d}{b^2} \left\{ (\gamma - 1)(b T_{L1} e^{b T_{L1}} - e^{b T_{L1}} + 1) + \delta \left( e^{b T_{L6}} - b T_{L6} - 1 \right) \right\} \]  \hspace{1cm} \text{(20)}

Let \( T_{L6} = T_{L1} + T_{L6} \), using \( I_{L1}(0) = I_{L5}(T_{L6}) \)

\[ e^{b T_{L1}} - 1 = \frac{\delta}{\gamma - 1} \left( e^{b T_{L6}} - 1 \right) \]  \hspace{1cm} \text{(21)}

The opportunity cost due to lost sales per cycle,

\[ C_o = \frac{C_3 (1 - \delta) a}{b} \left( e^{b T_{L6}} - 1 \right) \]  \hspace{1cm} \text{(22)}

Therefore, the total average cost per cycle for LIFO policy,

\[ K_L = \frac{1}{T_L} \left[ H_i + H_2 + C_D + C_S + C_o + C_i \right] \]

\[ = \left( \frac{h_i + C_1 \alpha t}{T} \right) \left( \gamma - 1 \right) a \left( \frac{T_{L2}^2}{2} + \frac{b}{6} T_{L2}^3 - \frac{\alpha}{12} T_{L2}^4 - \frac{\alpha b}{20} T_{L2}^5 \right) + W (T_{L3} + T_{L4}) \]
\[-\frac{\alpha W}{6}T_{L_3}^3 + a \left( \frac{1}{2}T_{L_5}^2 + \frac{b}{3}T_{L_3}^3 - \frac{\alpha b}{12}T_{L_5}^4 - \frac{\alpha b}{30}T_{L_5}^5 \right)\]

\[+ \frac{(h_2 + C_i \beta)}{T_L} \left( \frac{a}{b \beta(b+\beta)} \{ (\gamma-1)(b e^{\beta T_{L_3}} + b e^{\beta T_{L_3}} - b - \beta) - (b+\beta) e^{\beta T_{L_1}} + e^{(b+\beta)T_{L_3}} + \beta \} \right)\]

\[-\frac{\alpha W}{\beta^2} (e^{-\beta T_{L_3}} + \beta T_{L_3} - 1) \left( C_i a \frac{b}{b^2 T_L} \{(\gamma-1)(b T_{L_1} e^{\beta T_{L_3}} - e^{\beta T_{L_1}} + 1) \right.\]

\[+ \delta (e^{\beta T_{L_6}} - b T_{L_6} - 1) \left. + \frac{C_i (1-\delta) a}{b T_L} (e^{\beta T_{L_6}} - 1) + \frac{C_i}{T_L} = \frac{R}{T_L} \text{(say)} \right) \quad \text{...}(23)\]

Since \( I_{L_2} (T_{L_2}) = W, T_{L_2} \), which is a constant can be given by

\( \left( T_{L_2} + \frac{b}{2} T_{L_2} + \frac{\alpha}{6} T_{L_2}^3 \right) e^{\frac{\alpha}{2} T_{L_2}} = \frac{W}{a(\gamma-1)} \) \quad \text{...}(24)\]

Using \( I_{L_4} (T_{L_4}) = I_{L_5} (0) \) from Equations (4) and (5), we get \( T_{L_4} \) in terms of \( T_{L_3} \)

\( T_{L_4} = \frac{1}{b+\beta} \ln \left( \frac{(\gamma-1)(b e^{\beta T_{L_3}} - e^{-\beta T_{L_3}}) - \frac{\alpha W}{\beta} (1 - e^{-\beta T_{L_3}})}{a W \left( T_{L_5}^2 + \frac{b}{2} T_{L_5}^3 + \frac{\alpha}{6} T_{L_5}^3 \right)} \right) \) \quad \text{...}(25)\]

Also using \( I_{L_7} (0) = I_{L_6} (T_{L_4}) \) from Equations (6) and (7), where

\( T_{L_4}^2 = \frac{2}{\alpha} \ln \left[ \frac{a W \left( b T_{L_3} + \frac{b}{2} T_{L_3}^2 + \frac{\alpha}{6} T_{L_3}^3 \right)}{w \left( T_{L_5}^2 + \frac{b}{2} T_{L_5}^3 + \frac{\alpha}{6} T_{L_5}^3 \right)} \right] \) \quad \text{...}(26)\]

Therefore, the total cost per unit time can be expressed explicitly in terms of \( T_{L_3} \) and \( T_{L_5} \). The optimal value of \( T_{L_3} \) and \( T_{L_5} \) must satisfy the following two necessary conditions:

\[ \frac{\partial K_L}{\partial T_{L_3}} = 0 \quad \text{and} \quad \frac{\partial K_L}{\partial T_{L_5}} = 0 \]

Now that \( I_{L_4} (T_{L_4}) = -B_L \) and that \( I_{L_4} (T_{L_3}) = R_L \), we have

\[ B_L = \frac{(\gamma-1)a}{b} (e^{\beta T_{L_1}} - 1) \] \quad \text{...}(27)\]

and \( R_L = \frac{(\gamma-1)a}{b+\beta} (e^{\beta T_{L_3}} - e^{\beta T_{L_3}} - \frac{\alpha W}{b} (1 - e^{-\beta T_{L_3}}) \) \quad \text{...}(28)\]
There the optimal production policy can be easily derived after the optimal solutions $T_{L3}^*$ and $T_{LB}^*$ are obtained.

**Theorem:** Modified LIFO two-warehouse model always has a lower cost than Pakkala and Achary’s LIFO model if

$$\left( h_1 + C_i \alpha t \right) - \frac{\alpha}{\beta} \left( h_2 + C_i \beta \right) > 0.$$

**Proof:** Let $K_p$ be average total cost of Pakkala and Achary’s model (1994).

Let $T_{p1} = T - t_1$ and $T_{pi} = t_{i-1} - t_{i-2}$ for $i = 2, 3 ... 6$. After variables and parameters transformation, $K_p$ can expressed as

$$K_p = \frac{(h_1 + C_i \alpha t)}{T_p} \left[ (\gamma - 1) \alpha \left( \frac{T_{l2}^2}{2} + \frac{b}{T_{l2}^3} - \frac{\alpha}{12} T_{l2}^4 - \frac{\alpha b}{20} T_{l2}^5 \right) \right]$$

$$+ a \left[ \frac{1}{2} T_{l5}^2 + \frac{b}{3} T_{l5}^3 + \frac{\alpha}{8} T_{l5}^4 - \frac{\alpha}{24} T_{l5}^4 - \frac{\alpha b}{30} T_{l5}^5 \right]$$

$$+ \frac{(h_2 + C_i \beta)}{T_p} \left[ \frac{a}{b \beta (b + \beta)} \left( \gamma - 1 \right) \left( \beta e^{bT_{l1}} + b e^{-bT_{l3}} - b - \beta \right) \right]$$

$$- (b + \beta) e^{bT_{l4}} + \beta + \frac{C_i a}{b T_p} \left[ (\gamma - 1) \left( b T_{l1} e^{bT_{l1}} - e^{bT_{l1}} + 1 \right) \right] + \delta (e^{bT_{l6}} - b T_{l6} - 1)$$

$$+ \frac{C_i (1 - \delta)}{b} \left( e^{bT_{l6}} - 1 \right) + \frac{R}{T_p}$$

...(29)

For our convenience and without loss of generality, assuming that $K_{p_i} = K_{Li}$ for $i = 1, 2, ... 6$ for Equations (23) and (29), cost difference between modified LIFO and Pakkala and Achary’s LIFO model is given by:

$$K_L - K_p = \frac{(h_1 + C_i \alpha t)}{T_L} \left[ W(T_{l3} + T_{l4}) - \frac{\alpha W}{6} T_{l4}^3 \right]$$

$$+ \frac{(h_2 + C_i \beta)}{T_L} \left[ - \frac{\alpha W}{\beta} \left( e^{-\beta T_{l3}} + \beta T_{l3} - 1 \right) \right]$$

$$= \frac{W}{T_L} \left[ (h_1 + C_i \alpha t) \left( T_{l3} + T_{l4} \right) - \frac{\alpha T_{l4}^4}{6} \right]$$
$$-\frac{\alpha t(h_2 + C_i \beta)}{\beta}(e^{-\beta T_{L3}} + \beta T_{L3} - 1)$$  

...(30)

Since $WT_{L3} > 0$, if $\alpha$ is not significantly less than $\beta$, modified LIFO model will have a lower cost than Pakkala at Achary’s LIFO model under their assumption that $h_1 < h_2$.

4. FIFO Inventory Model

In FIFO dispatching policy, inventory items in OW that are stored first will first be released for consumption before that of RW. After the end of $T_{F3}$ (Figure 3), when production stops, inventory items in RW will remain in storage but with a deterioration rate $\beta$. Any demands are withdrawn from OW until the inventory items in OW are completely consumed, thereafter withdrawing from RW. Other inventory fluctuations and the decision objectives are all to be the same as those in a LIFO model.

![Figure 3: Inventory Level of Modified Two-Warehouse FIFO Model](image)

The differential equations describing inventory behavior for $I_{Fi}$, for $i = 1, 2$ and 8, are the same, as LIFO model and can be obtained from Equations (9), (10) and (16). Inventory level $I_{Fi}$, for $i = 3, \ldots, 7$ are described as follows:

$$I'_{F3} + \alpha t I_{F3}(t) = 0, \quad 0 \leq t \leq T_{F3} \quad \ldots (31)$$

$$I'_{F4}(t) + \beta I_{F4}(t) = P(t) - D(t), \quad 0 \leq t \leq T_{F3} \quad \ldots (32)$$
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\[ I_{F_2}'(t) + \beta I_{F_5}(t) = 0, \quad 0 \leq t \leq T_{F_4} \]  \hspace{1cm} (33)

\[ I_{F_6}'(t) + \alpha t I_{F_6}(t) = -D(t), \quad 0 \leq t \leq T_{F_4} \]  \hspace{1cm} (34)

\[ I_{F_7}'(t) + \beta I_{F_7}(t) = -D(t), \quad 0 \leq t \leq T_{F_5} \]  \hspace{1cm} (35)

Using the boundary conditions, \( I_{F_3}(0) = W, \ I_{F_4}(0) = 0, \ I_{F_5}(0) = I_{F_4}(T_{F_3}), \ I_{F_6}(T_{F_4}) = 0, \ I_{F_7}(T_{F_5}) = 0 \), the solutions of above equations can be obtained as:

\[ I_{F_3}(t) = We^{-at^2/2}, \quad 0 \leq t \leq T_{F_3} \]  \hspace{1cm} (36)

\[ I_{F_4}(t) = \left(\frac{y-1}{b+\beta}\right) e^{bt} - e^{-\beta t}, \quad 0 \leq t \leq T_{F_4} \]  \hspace{1cm} (37)

\[ I_{F_5}(t) = \left(\frac{y-1}{b+\beta}\right) e^{b(T_{F_3}-t)} - e^{-\beta t}, \quad 0 \leq t \leq T_{F_4} \]  \hspace{1cm} (38)

\[ I_{F_6}(t) = a \left[ (T_{F_4} - t) + \frac{b}{2} (T_{F_4}^2 - t^2) + \frac{a}{6} (T_{F_4}^3 - t^3) \right] e^{-at^2/2}, \quad 0 \leq t \leq T_{F_4} \]  \hspace{1cm} (39)

\[ I_{F_7}(t) = \frac{a}{b+\beta} \left[ e^{(b+\beta)T_{F_4} - \beta t} - e^{bt} \right], \quad 0 \leq t \leq T_{F_5} \]  \hspace{1cm} (40)

The inventory holding cost per cycle in OW and RW is:

\[ H_1 = h_1 \left[ \int_0^{T_{F_2}} I_{F_2}(t) \, dt + \int_0^{T_{F_3}} I_{F_3}(t) \, dt + \int_0^{T_{F_6}} I_{F_6}(t) \, dt \right] \]

\[ = h_1 \left[ (y-1) a \left( \frac{T_{F_2}^2}{2} + \frac{b}{6} T_{F_4}^3 - \frac{\alpha}{12} T_{F_4}^4 - \frac{\alpha b}{20} T_{F_4}^5 \right) + W \left( \frac{T_{F_3}^3}{6} - \frac{\alpha}{15} T_{F_4}^5 \right) \right] + a \left( \frac{T_{F_4}^2}{2} + \frac{b}{3} T_{F_4}^3 - \frac{\alpha}{12} T_{F_4}^4 - \frac{\alpha b}{15} T_{F_4}^5 \right) \]

and

\[ H_2 = h_2 \left[ \int_0^{T_{F_4}} I_{F_4}(t) \, dt + \int_0^{T_{F_5}} I_{F_5}(t) \, dt + \int_0^{T_{F_5}} I_{F_7}(t) \, dt \right] \]
\[
\frac{a h_2}{b \beta (b + \beta)} \left[ (\gamma - 1) \left( \beta e^{\beta T_3} + b e^{-\beta T_3} - \beta - b \right) 
\right. \\
+ (\gamma - 1) b \left( e^{\beta T_3} - e^{-\beta T_3} - e^{\beta T_3 - \beta T_3} + e^{-\beta (T_3 + T_4)} \right) \\
\left. + \left( e^{(b + \beta)T_5} - (b + \beta) e^{bT_5} + \beta \right) \right] 
\]

...(42)

The cost of deteriorated items per cycle

\[ C_D = C_1 \left\{ \frac{\alpha t}{h_2} H_1 + \frac{\beta}{h_2} H_2 \right\} \]

...(43)

Hence, the total cost per unit time of inventory system under FIFO dispatching policy is:

\[ K_F = \frac{1}{T_F} \left[ H_1 + H_2 + C_D + C_S + C_0 + C \right] \]

\[ = \frac{(h_1 + C_1 \alpha t)}{T_F} \left[ (\gamma - 1) a \left( \frac{T_{F_2}^2}{2} + \frac{b}{6} T_{F_2}^3 - \frac{\alpha}{12} T_{F_2}^4 - \frac{\alpha}{20} T_{F_2}^5 \right) 
\right. \\
\left. + W \left( T_{F_3} - \frac{\alpha T_{F_3}^3}{6} \right) + a \left( \frac{T_{F_4}^2}{2} + \frac{b}{3} T_{F_4}^3 + \frac{\alpha}{12} T_{F_4}^4 - \frac{\alpha}{15} T_{F_4}^5 \right) \right] 
\]

\[ + \frac{(h_2 + C_1 \beta)a}{T_F b \beta (b + \beta)} \left[ (\gamma - 1) \left( \beta e^{\beta T_3} + b e^{-\beta T_3} - \beta - b \right) 
\right. \\
\left. + (\gamma - 1) b \left( e^{\beta T_3} - e^{-\beta T_3} - e^{\beta T_3 - \beta T_3} + e^{-\beta (T_3 + T_4)} \right) 
\right. \\
\left. + \left( b e^{(b + \beta)T_5} - (b + \beta) e^{bT_5} + \beta \right) \right] 
\]

\[ + \frac{C_2 a}{b^2 T_F} \left[ (\gamma - 1) \left( b T_{F_1} e^{bT_1} - e^{bT_1} + 1 \right) + \delta e^{bT_6} - b T_{F_6} + 1 \right] 
\]

\[ + \frac{C_3 (1 - \delta) a}{b T_F} \left( e^{bT_6} - 1 \right) + \frac{C}{T_F} = \frac{R}{T_F} \text{ (Say)} \]

...(44)

In Equation (44), \( T_{F_2} \) is a constant that should have no difference with \( T_{L_2} \), i.e.,

\[ T_{F_2} = T_{L_2} \]
By using \( I_{F_3}(T_{F_3}) = I_{F_6}(0) \), we have a relation between \((T_{F_3} \text{ and } T_{F_4})\)

\[
\frac{\alpha T_{F_3}^2}{2} = a \left( T_{F_4} + \frac{b}{2} T_{F_4}^2 + \frac{\alpha}{6} T_{F_4}^3 \right) \quad \ldots(45)
\]

By using \( I_{F_3}(0) = I_{F_5}(T_{F_4}) \), we have

\[
T_{F_5} = \frac{1}{(b+\beta)} \ln \left[ 1 + (\gamma - 1) \left( e^{bT_{F_3} - \beta T_{F_4}} - e^{-\beta(T_{F_3} - T_{F_4})} \right) \right] \quad \ldots(46)
\]

Hence, the total average cost of the inventory cycle can be expressed in terms of variable \( T_{F_3} \) and \( T_{FB} \), where \( T_{FB} = T_{F_1} + T_{F_6} \). The optimal values of \( T_{F_3} \) and \( T_{FB} \) must satisfy the two necessary conditions \( \frac{\partial K}{\partial T_{F_3}} = 0 \) and \( \frac{\partial K}{\partial T_{FB}} = 0 \).

Furthermore, \( B \) can be derived as

\[
B_F = \frac{ae^{bt}}{\gamma a e^{bt} - a e^{bt}} T_{F_{FB}} \quad \ldots(47)
\]

and by using \( I_{F_4}(T_{F_3}) = R_F \), we have

\[
\Rightarrow R_F = \frac{(\gamma - 1) a}{(b+\beta)} \left( e^{bT_{F_3}} - e^{-\beta T_{F_3}} \right) \quad \ldots(48)
\]

Therefore, the optimal production policy under FIFO dispatching, i.e., \( B_F^* \) and \( R_F^* \), can also be derived after the optimal solutions \( T_{F_3}^* \) and \( T_{FB}^* \) are obtained.

**Particular Case:** If we apply the deterioration case \( \alpha \) of \( \beta \) in OW and RW, then the total average cost per cycle for LIFO policy, \( K_L \), is given by

\[
K_L = \frac{1}{T_L} \left[ H_1 + H_2 + C_D + C_S + C_0 + C' \right]
\]

\[
= \frac{(h_1 + C_1 \alpha t)}{T_L} \left[ \frac{a}{\alpha b (b+\alpha)} (\gamma - 1) \left( e^{bT_{L_2}} - e^{-\alpha T_{L_2}} - b - \alpha \right) + b e^{(b+\alpha)T_{L_5}} - (b+\alpha) e^{bT_{L_5}} + \alpha \right]
\]

\[
+ \left[ T_{L_3}^2 + \frac{1}{\alpha} \left( 1 - e^{-\alpha T_{L_3}} \right) \right] + \frac{h_2 + C_1 \beta}{T_L} \left[ \frac{a}{b \beta (b+\beta)} (\gamma - 1) \left( \beta e^{bT_{L_3}} + e^{-b\beta T_{L_3}} - b - \beta \right) \right]
\]
\[ + be^{(b+\beta)T_L} - (b + \beta)e^{bT_L} + \beta \right) - \frac{a W}{\beta^2}(e^{\beta T_L} + \beta T_L - 1) \]

\[ + \frac{C_2 a}{b^2 T_L} \left[ (\gamma - 1)(b T_L e^{b T_L} - e^{b T_L}) + 1 \right] \left( e^{\beta T_L} - b T_L - 1 \right) \]

\[ + \frac{C_3 (1 - \delta) a}{b T_L} \left( e^{b T_L} - 1 \right) + \frac{C_1}{T_L} = \frac{R}{T_L} \]

... (49)

Also, we find the values of

\[ T_{L_4} = \frac{1}{b + \beta} \ln \left\{ (\gamma - 1) \left( e^{b T_L} - e^{-\beta T_L} \right) - \frac{a W (b + \beta)}{\beta \alpha} \left( 1 - e^{-\beta T_L} \right) + 1 \right\} \]

... (50)

In this case, the total cost per unit time of inventory system under FIFO dispatching policy is:

\[ K_F = \frac{1}{T_F} \left[ H_1 + H_2 + C_D + C_S + C_0 + C' \right] \]

\[ = \left[ \frac{h_1 + C_1 \alpha}{T_F} \right] \left[ \frac{a}{b \alpha (b + \alpha)} \left( (b + \beta) \left\{ e^{b T_F} - e^{-\beta T_F} - b - \alpha \right\} + be^{(b+\alpha)T_F} \right) \right] \]

\[ + \frac{(h_2 + C_1 \beta) a}{b \beta (b + \beta) T_F} \left( (\gamma - 1) \left\{ e^{b T_F} - e^{-\beta T_F} + be^{(b + \alpha) T_F} \right\} \right) \]

\[ + \delta \left( e^{b T_F} - b T_F - 1 \right) \]

\[ + \frac{C_3 a (1 - \delta)}{b T_F} \left( e^{b T_F} - 1 \right) + \frac{C'}{T_F} = \frac{R}{T_F} \]

... (51)
By using \( I_{F3}(T_{F3}) = I_{F6}(0) \) and \( I_{F7}(0) = I_{F5}(T_{F4}) \), the values of \( T_{F4} \) and \( T_{F5} \) can be obtained respectively as

\[
T_{F4} = \frac{1}{b + \alpha} \ln \left[ 1 + \frac{(b + \alpha)W}{a} e^{-aT_{F3}} \right] \tag{52}
\]

and \( T_{F5} = \frac{1}{b + \alpha} \ln \left[ 1 + (\gamma - 1)\left(e^{bT_{F3}} - e^{-\beta T_{F3}}\right)e^{-\beta T_{F4}} \right] \tag{53} \]

**Theorem 1:** Modified LIFO two-warehouse model always has a lower cost than Pakkala and Achary’s LIFO model if \( \left(h_1 + C_1\alpha\right) > \left(h_2 + C_1\beta\right) \times \frac{\alpha}{b} \).

**Proof:** Suppose \( K_p \) as average total cost of Pakkala and Achary’s (1994) model. If \( T_{P1} = T - t_1 \) and \( T_{Pj} = T - t_{i-1} - t_{i-2} \) for \( i = 2, 3 \ldots 6 \). After variables and parameters transformation, Pakkala and Achary’s Equation (11) can be expressed as:

\[
K_p = \left(\frac{h_1 + C_1\alpha}{T_p}\right) \left[ \frac{a}{\alpha b(b + \alpha)} \left( (\gamma - 1)\left(\alpha e^{bT_{P3}} + be^{-\alpha T_{P2}} - b - \alpha \right) \right. \\
+ be^{(b+a)T_{P5}} - (b + \alpha)e^{bT_{P5}} + \alpha \left) \right. \\
+ \left(\frac{h_2 + C_1\beta}{T_p}\right) \left[ \frac{a}{b \beta (b + \beta)} \left( (\gamma - 1)\left(\beta e^{bT_{P3}} + be^{-\beta T_{P3}} - b - \beta \right) \right. \\
+ be^{(b+\beta)T_{P3}} - (b + \beta)e^{bT_{P4}} + \beta \left) \right. \\
+ \frac{C_2 a}{b^2 T_p} \left[ (\gamma - 1)(b T_{P1} e^{bT_{P1}} - e^{bT_{P1}} + 1) + \delta(e^{bT_{P1}} - b T_{P1} - 1) \right] \\
+ \frac{C_3 (1 - \delta) a}{b T_p} (e^{bT_{P1}} - 1) + \frac{R}{P} \right) \tag{54}
\]

For our convenience and without loss of generality, we assume that \( T_{P1} = T_{Li} \) for \( i = 1, 2 \ldots 6 \). The cost difference between modified LIFO and Pakkala and Achary’s LIFO model is given by:
Since $WT_{L_3} > 0$, and $\alpha$ is not significantly less than $\beta$, modified LIFO model will have a lower cost than Pakkala and Achary's LIFO model under their assumption that $h_1 < h_2$.

**Theorem 2:** If the two warehouses have the same deterioration date, i.e., $\alpha = \beta$, then $K_F > K_L$ if $h_1 < h_2$; otherwise $K_F < K_L$ if $h_1 > h_2$.

**Proof:** Let $T_3$ and $T_B$ be the decision objectives of the two models. We want to prove that if $h_1 < h_2$ and $\alpha = \beta$, then $K_F < K_L$ for any combinations of $K(T_3, T_B)$. First, let $T_{L_3} = T_{F_3}, T_{L_B} = T_{F_B}$. Then, we have

$$K_F - K_L = \frac{a}{T_F} \left[ \frac{(h_2 + C_i \alpha)\alpha}{b} - (h_1 + C_i \alpha) \frac{WT_{L_3}}{a} + (h_2 + C_i \alpha) \right]$$

$$+ \left[ \frac{1}{\alpha} \left( T_{F_3} - T_{L_3} \right) + \frac{1}{ab} \left( e^{-\alpha T_{L_3}} - 1 \right) \right] > 0 \quad \ldots(56)$$

This theorem implies that, in facing policy choice, if the two warehouses have similar preservation conditions that the inventory deterioration are nearly the same, then the policy choice sale depends on the difference in inventory holding cost, namely $h_1$ and $h_2$. FIFO policy will be less expensive when $h_1 < h_2$, otherwise, LIFO is suggested. Undoubtedly, when the two warehouses have the same parameters in all respects, i.e., $\alpha = \beta$ and $h_1 = h_2$, these two policies should have no difference, i.e., $K_F = K_L$.

**Conclusion**

Most important for managers that deal with perishable products, using FIFO, rather than LIFO, is a common accepted practice of making sure that the products are dispatched at its maximum freshness. In this paper, a two-warehouse inventory model for deteriorating inventory items with FIFO dispatching policy has been proposed. It has been proven that when the deterioration rate is different in the two warehouses, FIFO is less expensive than LIFO, provided the holding cost in RW is lower than OW. In addition, Pakkala and Achary’s two-warehouse LIFO model has been sufficiently modified to be more complete.
References


Reference # 61J-2009-09-01-01