Calderon Type Reproducing Formula Associated with the Bessel Type Differential Operator

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In this paper, generalized convolution # on the half-line corresponding to the Bessel type differential operator is defined. A Calderon type reproducing formula associated with generalized convolution involving finite Borel measure, which leads to results on the $L^p$-norm and point-wise approximation for functions on the half-line, is studied.

**Keywords:** Calderon reproducing formula, Generalized convolution, Fourier-Bessel transform, Bessel type differential operator, Plancherel type formula

**Introduction**

The classical Calderon reproducing formula Calderon (1964) can be formulated as follows. Let $g$ and $h$ be $L^2$—functions on $\mathbb{R}$ satisfying the condition

$$\int_0^\infty G(a\lambda)H(a\lambda)\frac{da}{a} = 1, \text{ for all } \lambda \in \mathbb{R} - \{0\}$$

where $G$ and $H$ respectively denote the classical Fourier transforms of $g$ and $h$ on $\mathbb{R}$.

Set

$$g_a(x) = \frac{1}{a} g\left(\frac{x}{a}\right), \quad h_a(x) = \frac{1}{a} h\left(\frac{x}{a}\right)$$

Then

$$f = \int_0^\infty f^{*} g_a^{*} h_a \frac{da}{a}$$

...(1)

where * denotes the usual convolution product on $\mathbb{R}$.

Formula (1) was originally used in the so-called Calderon-Zygmund theory of singular integral operators later, it was extended to various areas of applied mathematics, particularly, in wavelet theory by Daubechies (1992); and Frazier et al., (1991).

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If $\mu$ is a finite Borel measure on the real line $\mathbb{R}$, then (1) has a natural generalization to:

$$f = \int_0^a f^* \frac{d\mu}{a},$$  \hspace{1cm} \text{(2)}

where $\mu_a$ is the dilated measure of $\mu$. Under some restrictions on $\mu$, the $L^p$-norm or a.e. convergence of (2) has been proved by Rubin and Shamir (1995). A more general form of (2) has been investigated by Rubin (1996, Sect. 12).

The aim of the paper is to study similar questions when in (2) the classical convolution $*$ is replaced by generalized convolution $#$ on the half-line generated by the Bessel type differential operator

$$\Delta_{\alpha, \beta} = D^2 + \frac{2(2\alpha - \beta + 1)}{x} D, \quad (\alpha - \beta) > -1,$$  \hspace{1cm} \text{(3)}

where $D = \frac{d}{dx}$

It is believed that Calderon’s reproducing formula as discussed here will be of great utility in Inversion Problems (Mourou and Trimeche, 1996; Trimeche, 1995 and 1997) and in Wavelet theory on Bessel-Kingman hypergroups (Trimeche, 1996). The major instrument of this extension is the Fourier-Bessel transform and related harmonic analysis results. Some facts about the Littlewood-Paley theory for this transform are used (Stempak, 1986; and Xu, 1995).

### 2. Preliminaries

In this section, we recall some basic results in harmonic analysis related to the Fourier-Bessel type transform. For more details refer to Trimeche (1981 and 1997).

We define $L^p\left(x^{2(\alpha-\beta+1)}dx\right), 1 \leq p < \infty$ as the class of measurable functions $f$ on $[0, \infty)$ for which

$$\|f\|_{p, \alpha, \beta} = \left[\int_0^a |f(x)|^p x^{2(\alpha-\beta+1)} \, dx\right]^\frac{1}{p}, \text{ if } 1 \leq p < \infty$$

$$\|f\|_{\infty, \alpha, \beta} = \sup_{x \geq 0} |f(x)|$$

$B = B([0, \infty))$ denotes the space of finite Borel measures on the half-line $[0, \infty)$.

For each $\mu \in B$, write $\|\mu\| = |\mu|([0, \infty))$, where $|\mu|$ is the absolute value of $\mu$. 

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**Definition 1:** The Fourier-Bessel type transform of a measure $\mu \in \mathcal{B}$ is defined by:

$$FB[\mu(\lambda)] = \int_{[0, \infty)} j_{a/\beta + 1/2} (\lambda, x) \, d\mu(x), \lambda \geq 0$$

where

$$j_{a/\beta + 1/2} = 2^{a/\beta + 1/2} \Gamma\left(\alpha - \beta + \frac{3}{2}\right) s^{(a/\beta + 1/2)} J_{a/\beta + 1/2}(s)$$

with $J_v$ the Bessel function of the first kind and order $v$.

In view of Schwartz (1971), we have the following properties:

**Properties 1**

- For all $\mu \in \mathcal{B}$, the function $FB(\mu)$ is continuous on $[0, \infty)$ and

$$\lim_{\lambda \to \infty} FB\mu(\lambda) = \mu([0]) \quad \ldots \quad (4)$$

- The Fourier-Bessel type transform $FB$ maps injectively $B$ into $C_b([0, \infty))$ (the space of continuous and bounded functions on $[0, \infty)$).

An outstanding result for the Fourier-Bessel type transform is the following Plancherel type theorem.

**Theorem 1**

- For every $f \in L^1 \cap L^2 \left(x^{2(a-\beta+1)} \, dx\right)$, we have the Plancherel type formula

$$\int_0^\infty |f(x)|^2 x^{2(a-\beta+1)} \, dx = \int_0^\infty |FB(f)(\lambda)|^2 \frac{\lambda^{2(a-\beta+1)}}{2^{2(a-\beta+1/2)}(\Gamma(\alpha - \beta + 3/2))^2} \, d\lambda$$

- The Fourier-Bessel type transform $FB$ extends uniquely to a unitary isomorphism from $L^2 \left(x^{2(a-\beta+1)} \, dx\right)$ onto $L^2 \left(\lambda^{2(a-\beta+1)} / 2^{(a-\beta+1/2)} \cdot \Gamma\left(\alpha - \beta + \frac{3}{2}\right)^2\right) \, d\lambda$ the inverse transform is given by

$$\left(FB\right)^{-1} = \frac{1}{2^{2(a-\beta+1)} \left(\Gamma\left(\alpha - \beta + \frac{3}{2}\right)^2\right)} FB$$
Definition 2

• The generalized translation type operators $T_x, x \geq 0$ are defined for smooth functions on $[0, \infty)$ by

$$T_x(f)(y) = \frac{\Gamma \left( \alpha - \beta + \frac{3}{2} \right)}{\sqrt{\pi} \Gamma (\alpha - \beta + 1)} \int_0^\infty f \left( \sqrt{x^2 + y^2 + 2xy \cos \theta} \right) (\sin \theta)^{2(\alpha - \beta + 1)} d\theta$$

• The generalized convolution type product of a measure $\mu \in B$ and a smooth function $f$ on $[0, \infty)$ are defined by

$$\mu \ast f(x) = \int_{[0, \infty)} T_x f(y) d\mu(y), \ x \geq 0$$

From Mourou and Trimeche (1996) and Trimeche (1997), we have the following properties:

Properties 2

• The functions $j_{\alpha - \beta + \frac{1}{2}}(\lambda \bullet), \lambda \in \mathbb{C}$ satisfy on $[0, \infty)$ the product formula

$$j_{\alpha - \beta + \frac{1}{2}}(\lambda x) j_{\alpha - \beta + \frac{1}{2}}(\lambda y) = T_x \left( j_{\alpha - \beta + \frac{1}{2}}(\lambda \bullet) \right)(y)$$

• Let $f \in L^p \left( x^{2(\alpha - \beta + 1)} dx \right), 1 \leq p < \infty$. Then for all $x \geq 0$ the function

$$T_x f \in L^p \left( x^{2(\alpha - \beta + 1)} dx \right) \text{ and } \|T_x f\|_{p, \alpha, \beta} \leq \|f\|_{p, \alpha, \beta}.$$  

• For $f \in L^p \left( x^{2(\alpha - \beta + 1)} dx \right) p = 1 \text{ or } 2 \text{ and } x \geq 0$, we have

$$FB(T_x f)(\lambda) = j_{\alpha - \beta + \frac{1}{2}}(\lambda x) FB(f)(\lambda)$$

• If $\mu \in B$ and $f \in L^p \left( x^{2(\alpha - \beta + 1)} dx \right), 1 \leq p \leq \infty$ then $\mu \ast f \in L^p \left( x^{2(\alpha - \beta + 1)} dx \right)$ and

$$\|\mu \ast f\|_{p, \alpha, \beta} \leq \|\mu\| \|f\|_{p, \alpha, \beta},$$  

... (5)

• For $\mu \in B$ and $f \in L^p \left( x^{2(\alpha - \beta + 1)} dx \right) p = 1 \text{ or } 2$, we have

$$FB(\mu \ast f) = FB(\mu) FB(f),$$  

... (6)
Definition 3: Let $\mu \in \mathbb{B}$ and $a > 0$. We define the dilated measure $\mu_a$ of $\mu$ by

$$
\int_{[0, \infty)} \phi(x) d\mu_a(x) = \int_{[0, \infty)} \phi(ax) d\mu(x), \phi \in C_c([0, \infty))
$$

where, $C_c([0, \infty))$ denotes the space of continuous functions on $[0, \infty)$ with compact support.

Properties 3

- When $\mu = g(x)x^{2(\alpha - \beta + 1)} dx$, with $g \in L^1(x^{2(\alpha - \beta + 1)} dx)$, the measure $\mu_a, a > 0$ is given by the function

$$
g_a(x) = \frac{1}{a^{2(\alpha - \beta + 1)+1}} g\left(\frac{x}{a}\right), \quad x \geq 0 \quad \text{(7)}
$$

- Let $\mu \in \mathbb{B}$ and $a > 0$. Then $FB\mu_a(\lambda) = FB\mu(a\lambda)$ for all $\lambda \geq 0$. \( \text{(8)} \)

- For $\mu \in \mathbb{B}$ and $f \in L^p(x^{2(\alpha - \beta + 1)} dx), 1 \leq p < \infty$, we have

$$
\lim_{a \to 0} (\mu_a \# f) = \mu \left( [0, \infty) \right) f, \quad \text{(9)}
$$

where the limit is in $L^p(x^{2(\alpha - \beta + 1)} dx)$.

- Let $g \in L^1(x^{2(\alpha - \beta + 1)} dx)$ and $f \in L^p(x^{2(\alpha - \beta + 1)} dx), 1 < p < \infty$.

Then

$$
\lim_{a \to \infty} f \# g_a = 0, \quad \text{(10)}
$$

where the limit is in $L^p(x^{2(\alpha - \beta + 1)} dx)$.

3. Calderon’s Formula Associated with the Bessel Type Operator $\Delta_{\alpha,\beta}$

In this section, we define analogues to Equation (2) for the generalized convolution $\#$ and investigate its convergence in the $L^p$-norm or point-wise sense. For this, we need the following lemmas:

**Lemma 1:**

Let $\mu \in \mathbb{B}$. For $0 < \delta < \infty$, define

$$
G_{\varepsilon, \delta}(x) = \frac{\mu\left(\left[\frac{x}{\varepsilon}, \frac{x}{\delta}\right]\right)}{x^{2(\alpha - \beta + 1)+1}}, \quad x > 0 \quad \text{(11)}
$$
and

\[ K_{\varepsilon, \delta}(\lambda) = \int_{-\infty}^{\infty} FB\mu\left(a\lambda\right) \frac{da}{a}, \quad \lambda \geq 0 \quad \text{...(12)} \]

Then \( G_{\varepsilon, \delta} \in L^1\left(x^{2(a-\beta+1)}dx\right) \) and

\[ FB\left(G_{\varepsilon, \delta}\right) = K_{\varepsilon, \delta} - \mu\left(\{0\}\right) \log\left(\frac{\delta}{\epsilon}\right) \quad \text{...(13)} \]

**Proof**

\[ \int_{0}^{\infty} \left| G_{\varepsilon, \delta}(x) \right| x^{2(a-\beta+1)} dx = \int_{0}^{\infty} \left| \mu\left(\frac{x}{\delta}, \frac{\gamma}{\epsilon}\right) \right| x^{2(a-\beta+1)} dx \]

\[ \leq \int_{(0, \infty)} \left( \int d\mu\left( y \right) \right) \frac{dy}{y} \]

\[ = \log\left(\frac{\delta}{\epsilon}\right) \mu\left(0, \infty\right) < \infty \]

Using Fubini’s theorem, we obtain

\[ FB\left(G_{\varepsilon, \delta}\right)(\lambda) = \int_{(0, \infty)} \left( \int d\mu\left( y \right) \right) \int_{-\infty}^{\infty} j_{a-\beta+\frac{1}{2}}(\lambda x) \frac{dx}{x} \]

\[ = \int_{(0, \infty)} \left( \int j_{a-\beta+\frac{1}{2}}(\lambda x) \frac{dx}{x} \right) d\mu\left( y \right) \]

\[ = \int_{(0, \infty)} \left( \int j_{a-\beta+\frac{1}{2}}(y \lambda x) \frac{dx}{x} \right) d\mu\left( y \right) \]

\[ = \int_{(0, \infty)} \left( \int j_{a-\beta+\frac{1}{2}}(y \lambda x) \frac{dx}{x} \right) d\mu\left( y \right) \]
\[
\int_{(\varepsilon)} \left[ F \mu (\lambda x) - \mu \{0\} \right] \frac{dx}{x} = K_{\varepsilon, \delta} (\lambda) - \mu \{0\} \log \left( \frac{\delta}{\varepsilon} \right)
\]

**Lemma 2:**

Let \( \mu \in \mathbb{B} \). Then for \( f \in L^p \left( x^{2(\alpha - \beta + 1)} \right), 1 \leq p \leq \infty \) and \( 0 < \varepsilon < \delta < \infty \) the function

\[
f^{\varepsilon, \delta} (x) = \int_{\varepsilon}^{\delta} f \# \mu_x (x) \frac{da}{a}
\]

belongs to \( L^p \left( x^{2(\alpha - \beta + 1)} \right) \) and has the form

\[
f^{\varepsilon, \delta} (x) = f \# G_{\varepsilon, \delta} (x) + \mu \{0\} f (x) \log \left( \frac{\delta}{\varepsilon} \right),
\]

where \( G_{\varepsilon, \delta} \) is given in Equation (11).

**Proof:** We have

\[
f^{\varepsilon, \delta} (x) = \int_{\varepsilon}^{\delta} f \# \mu_x (x) \frac{da}{a}
\]

\[
= \int_{\varepsilon}^{\delta} \left( \int_{(0, \infty)} T_x f (ay) d\mu (y) \frac{da}{a} \right)
\]

By Fubini’s theorem, we have

\[
= \int_{(0, \infty)} \left( \int_{\varepsilon}^{\delta} T_x f (ay) \frac{da}{a} \right) d\mu (y)
\]

\[
= \mu \{0\} f (x) \log \left( \frac{\delta}{\varepsilon} \right) + \int_{(0, \infty)} \left( \int_{\varepsilon}^{\delta} T_x f (ay) \frac{da}{a} \right) d\mu (y)
\]

\[
= \mu \{0\} f (x) \log \left( \frac{\delta}{\varepsilon} \right) + \int_{(0, \infty)} T_x f (a) \left( \int_{\varepsilon}^{\delta} \frac{d\mu (y)}{\delta^2 \varepsilon^2} \right) \frac{da}{a}
\]
Now, by using the above relation inequality (5) and Lemma 1, it is clear that
\[ f^{\varepsilon, \delta} \in L^p \left( x^{2(a-\beta+1)} \right) . \]

In the sequel, it will be convenient to investigate separately the \( L^2 \) and \( L^p \) convergence of the truncated integrals in Equation (14).

**Lemma 3:**

Let \( \mu \in B \). Then for \( f \in L^2 \left( x^{2(a-\beta+1)} \right) \) we have
\[
FB \left( f^{\varepsilon, \delta} \right) = FB \left( f \right) K_{\varepsilon, \delta}
\]
where \( K_{\varepsilon, \delta} \) is the function defined in (12).

**Proof:** From Equation (15), we have
\[
f^{\varepsilon, \delta} (x) = f \# G_{\varepsilon, \delta} (x) + \mu \left( \{0\} \right) f (x) \log \left( \frac{\delta}{\varepsilon} \right)
\]
Therefore,
\[
FB \left( f^{\varepsilon, \delta} (x) \right) = FB \left( f \right) FB \left( G_{\varepsilon, \delta} (x) \right) + FB \left[ \mu \left( \{0\} \right) f (x) \log \left( \frac{\delta}{\varepsilon} \right) \right]
\]
By using Equation (6), we can obtain
\[
FB \left( f^{\varepsilon, \delta} (x) \right) = FB \left( f \right) FB \left( G_{\varepsilon, \delta} (x) \right) + \mu \left( \{0\} \right) \log \left( \frac{\delta}{\varepsilon} \right) FB \left( f \right)
\]
By using Equation (13), we can have
\[
FB \left( f^{\varepsilon, \delta} (x) \right) = FB \left( f \right) K_{\varepsilon, \delta} - FB \left( f \right) \mu \left( \{0\} \right) \log \left( \frac{\delta}{\varepsilon} \right) + \mu \left( \{0\} \right) \log \left( \frac{\delta}{\varepsilon} \right) FB \left( f \right) = FB \left( f \right) K_{\varepsilon, \delta}
\]

**Theorem 2:**

Let \( \mu \in B \) be such that the integral
\[
A_{\mu} = \int_{0}^{\infty} FB \left( \mu \left( \lambda \right) \right) \frac{d\lambda}{\lambda} < \infty
\]
Then for all \( f \in L^2 \left( x^{2(a-\beta+1)} \right) \) we have
\[
\lim_{\delta \to 0} \left\| f^{\varepsilon, \delta} - A_\mu f \right\|_{2, \alpha, \beta} = 0. \quad \text{...(18)}
\]

**Proof:** By definition

\[
\left\| f^{\varepsilon, \delta} - A_\mu f \right\|_{2, \alpha, \beta}^2 = \int_0^\infty \left| f^{\varepsilon, \delta} - A_\mu f \right|^2 x^{2 \alpha} dx
\]

We make use of Theorem 1 to obtain

\[
\left\| f^{\varepsilon, \delta} - A_\mu f \right\|_{2, \alpha, \beta}^2 = \int_0^\infty \left| FB\left( f^{\varepsilon, \delta} - A_\mu f \right) \right|^2 \frac{x^{2 \alpha}}{2^{2 \alpha} \left( \Gamma \left( \alpha - \beta + \frac{3}{2} \right) \right)^2} d\lambda
\]

\[
= \frac{1}{2^{2 \alpha} \left( \Gamma \left( \alpha - \beta + \frac{3}{2} \right) \right)^2} \int_0^\infty \left| FB\left( f^{\varepsilon, \delta} - A_\mu f \right) (\lambda) \right|^2 x^{2 \alpha} d\lambda
\]

\[
= \frac{1}{2^{2 \alpha} \left( \Gamma \left( \alpha - \beta + \frac{3}{2} \right) \right)^2} \left\| FB\left( f^{\varepsilon, \delta} - A_\mu f \right) \right\|_{2, \alpha, \beta}^2
\]

Now, by using Equation (16), we obtain

\[
\left\| f^{\varepsilon, \delta} - A_\mu f \right\|_{2, \alpha, \beta}^2 = \frac{1}{2^{2 \alpha} \left( \Gamma \left( \alpha - \beta + \frac{3}{2} \right) \right)^2} \left\| FB\left( f^{\varepsilon, \delta} - A_\mu f \right) \right\|_{2, \alpha, \beta}^2
\]

From Equations (12) and (17) and dominated convergence theorem, it is clear that

\[
\left\| f^{\varepsilon, \delta} - A_\mu f \right\| \to 0 \quad \text{as} \quad \varepsilon \to 0, \delta \to \infty
\]

That is

\[
\lim_{\delta \to 0} \left\| f^{\varepsilon, \delta} - A_\mu f \right\|_{2, \alpha, \beta} = 0
\]

**Remark:** The condition \( FB(\mu)(0) = 0 \), which is equivalent to \( \mu \left( [0, \infty) \right) = 0 \) is necessary for the convergence of \( A_\mu \).
Lemma 4:

Let \( \mu \in B \) be such that

\[
\int_0^\infty |\mu([0,y])| \frac{dy}{y} < \infty.
\] ...(19)

The integral \( A_\mu \) is finite and admits the representation

\[
A_\mu = \int_0^\infty \mu([0,y]) \frac{dy}{y}
\] ...(20)

Proof: Note that if

\[
G(y) = \frac{\mu([0,y])}{y^{2(\alpha-\beta+1)}}, \quad y > 0
\] ...(21)

then from Equation (11), we have

\[
G_\varepsilon, \delta = G_\varepsilon - G_\delta
\] ...(22)

where \( G_\varepsilon, G_\delta \) are the dilated functions of \( G \).

As \( G \in L^p(\chi^{2(\alpha-\beta+1)} \, dx) \), we can use Equations (8) and (13) to obtain

\[
FB(G_{\varepsilon, \delta})(\lambda) = \int_{\varepsilon \lambda}^{\delta \lambda} FB\mu(a) \, da - \mu([0]) \log \left( \frac{\delta}{\varepsilon} \right) = FB G(\varepsilon \lambda) - FB G(\delta \lambda)
\] ...(23)

for all \( \lambda > 0 \). Now, Equation (19) implies necessarily \( \mu([0]) = 0 \).

Hence, when \( \varepsilon = 1 \) and \( \delta \to \infty \), a combination of Equations (23) and (11) gives

\[
FB G(\lambda) = \int_\lambda^\infty FB\mu(a) \, da
\] for all \( \lambda > 0 \) ...(24)

By using continuity of \( FB(\mu) \), formula Equation (20) follows from Equation (24).

Theorem 3:

Let \( \mu \in B \) satisfy the condition of Lemma 4, and \( f \in L^p(\chi^{2(\alpha-\beta+1)} \, dx) \), \( 1 < p < \infty \). Then

\[
\lim_{\varepsilon, \delta \to 0^+} f^{\varepsilon, \delta} = A_\mu f
\] ...(25)

where the limit is in \( L^p(\chi^{2(\alpha-\beta+1)} \, dx) \).

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Proof: By Equations (15) and (22) we have

\[ f^{e, \delta}(x) = f \# G_{e, \delta}(x) + \mu(\{0\}) f(x) \log \left( \frac{\delta}{e} \right) \]

\[ = f \# G_e - f \# G_\delta \quad \text{(since } \mu(\{0\}) = 0) \quad \text{...}(26) \]

Equation 25 follows from Equations (9), (10) and (20).

4. Point-wise Convergence of Equation 14

The point-wise convergence of Equation (14) involves some detailed study from the Littlewood-Paley theory for the Fourier-Bessel type transform (Achour and Trimeche, 1983; Stempak, 1986; and Xu, 1995).

Definition 4: Let \( f \) be a locally integrable function on \([0, \infty)\). We call \( x \in [0, \infty) \) a Lebesgue point of \( f \) if

\[ \lim_{\varepsilon \to 0} \frac{1}{t^{2(a-\beta+1)} \varepsilon} \int_0^t \left| T_x f(y) - f(x) \right| y^{2(a-\beta+1)} dy = 0 \]

By referring to Stempak (1986) and Xu (1995), we have the following properties:

Properties 4

• If \( f \) is continuous at \( x_0 \), then \( x_0 \) is Lebesgue point of \( f \).

• If \( f \) is locally integrable, then for almost every \( x \in [0, \infty) \), \( x \) is a Lebesgue point of \( f \).

• Let \( g \) be a measurable function on \([0, \infty)\) such that \( \text{ess sup}_{t \geq 1} |g(t)| \in L^p \left( x^{2(a-\beta+1)} dx \right) \). Let, \( f \in L^p \left( x^{2(a-\beta+1)} dx \right), 1 \leq p \leq \infty \). Then for each Lebesgue point \( x \) of \( f \).

\[ \lim_{\varepsilon \to 0} f \# g_\varepsilon(x) = f(x) \int_0^x g(t) t^{2(a-\beta+1)} dt \quad \text{...}(27) \]

where \( T_x \) is as in (7).

Theorem 4:

Let \( \mu \in B \) satisfying

\[ \int_0^1 |\mu| \left( \left[ 0, y \right] \right) \frac{dy}{y} < \infty \quad \text{...}(28) \]
and
\[
\int_{0}^{\infty} \left| \mu \left( [0, y] \right) \right| \frac{dy}{y} < \infty
\] ...(29)
and let \( f \in L^{p} \left( x^{2(a-\beta+1)} \, dx \right), 1 \leq p < \infty \).

If \( \text{ess sup}_{t \geq 1} |G(t)| \in L^{1} \left( x^{2(a-\beta+1)} \, dx \right) \), then for each Lebesgue point \( x \) of \( f \)
\[
\lim_{\delta \to 0} f^{x, \delta} (x) = \lambda_{\mu} f(x)
\]

**Proof:** By Equations (20), (26) and (27), it is sufficient to prove that \( f \# G_{\delta}(x) \to 0 \) as \( \delta \to \infty \).

Note that as \( \frac{|\mu((0, y])|}{y} \to |\mu((0, y])| \), as \( y \to 0 \), Equation (28) implies necessarily \( \mu((0]) = 0 \). We have
\[
f \# G_{\delta}(x) = \int_{0}^{\infty} T_{x} f(y) \mu \left( \left[ 0, \frac{y}{\delta} \right] \right) \frac{dy}{y}
\]
\[
= \int_{0}^{1} T_{x} f(y) \mu \left( \left[ 0, \frac{y}{\delta} \right] \right) \frac{dy}{y} + \int_{1}^{\infty} T_{x} f(y) \mu \left( \left[ 0, \frac{y}{\delta} \right] \right) \frac{dy}{y}
\]
\[
= M_{\delta}(x) + N_{\delta}(x),
\]
where,
\[
M_{\delta}(x) = \int_{0}^{1} T_{x} f(y) \mu \left( \left[ 0, \frac{y}{\delta} \right] \right) \frac{dy}{y}
\]
and
\[
N_{\delta}(x) = \int_{1}^{\infty} T_{x} f(y) \mu \left( \left[ 0, \frac{y}{\delta} \right] \right) \frac{dy}{y}
\]
For \( \delta \geq 1 \), we have
\[
|T_{x} f(y) \mu \left( \left[ 0, \frac{y}{\delta} \right] \right)| / y \leq T_{x} |f(y)| \mu \left( \left[ 0, y \right] \right) / y
\]
and

\[ \int_0^1 T_x \left| f\left( y \right) \right| \mu \left( \left[ 0, y \right] \right) \frac{dy}{y} < \infty \]

By using Equation (28) and the dominated convergence theorem, we can infer that

\[ \lim_{\delta \to \infty} M_\delta(x) = 0 \]

For \( 1 \leq p < \infty \), the function \( y \to T_x f(y)/y \) is integral on \([0, \infty)\), by (29) and again by dominated convergence theorem, it follows that:

\[ \lim_{\delta \to \infty} N_\delta(x) = 0 \]

**Lemma 5:**

Let \( f \) be an essentially bounded function on \([0, \infty)\) that is weakly oscillating around 0. That is

\[ \lim_{t \to \infty} \frac{1}{t^{2(\alpha-\beta+1)+1}} \int_0^t T_x f(y) y^{2(\alpha-\beta+1)} dy = 0, \text{ for all } x \geq 0. \] \[ \text{(30)} \]

Let \( g \in L^1(x^{2(\alpha-\beta+1)} dx) \).

Then,

\[ \lim_{a \to \infty} f \# g_a(x) = 0 \]
uniformly for \( x \in [0, \infty) \).

**Proof:** It is sufficient to consider the case where \( f \geq 0 \) and \( g \) continuous with support in \([0, R] \). Then by virtue of (30).

\[ \left| f \# g_a(x) \right| = \left| \int_0^R T_x f(y) g_a(y) y^{2(\alpha-\beta+1)} dy \right| \]

\[ \leq R^{2(\alpha-\beta+1)} \left( \frac{1}{(aR)^{2(\alpha-\beta+1)+1}} \int_0^a T_x f(y) y^{2(\alpha-\beta+1)} dy \right) \|g\|_\infty \to 0 \text{ as } a \to \infty \]

This implies that \( \lim_{a \to \infty} f \# g_a(x) = 0 \).
Theorem 5:

Let $\mu \in \mathcal{B}$ satisfy Equation (19) and let $f \in L^\infty([0, \infty), dx)$ be weakly oscillating around (Equation 30).

If $\sup_{t \geq x} |G(t)| \in L^1(\chi^{2(\alpha-\beta+1)}dx)$ (Equation 21), then in any Lebesgue point $x$ of $f$ we have

$$\lim_{\delta \to 0} f^{\epsilon, \delta}(x) = A_\mu f(x)$$

Proof: Proof follows from Equations (20), (26) and (27) and Lemma 5.

It is interesting to express the integral $A_\mu$ in Equation (17) just in terms of $\mu$ without the Fourier-Bessel transform. To know under what restrictions on the measure $\mu$ this holds, we need the following simple Lemma.

Lemma 6:

Let $\mu \in \mathcal{B}$ then the following two assumptions on $\mu$ are equivalent:

1. $\int_0^1 |\mu|[([0, y])](dy/y) < \infty$ and $\int_1^\infty |\mu|[((y, \infty))](dy/y) < \infty$

2. $\int_{[0, \infty)} \log x |d\mu|(x) < \infty$

Proof: Proof follows from the relations

$$\int_0^1 |\mu|[([0, y])]\frac{dy}{y} = -\int_{[0, 1]} \log x |d\mu|(x)$$

and

$$\int_1^\infty |\mu|[((y, \infty))]\frac{dy}{y} = \int_{[1, \infty]} \log x |d\mu|(x)$$

Theorem 6:

Let $\mu \in \mathcal{B}$ satisfy the condition

$$\int_{[0, \infty)} \log x |d\mu|(x) < \infty$$

...(33)
Then the integral $A_\mu$ is finite and represented as:

$$A_\mu = \int_{[0, \infty)} \log \left( \frac{1}{x} \right) d \mu(x)$$

...(34)

**Proof:** By Lemmas 4 and 6, the integral $A_\mu$ exists and has the expression

$$A_\mu = \int_0^\infty \mu([0, y]) \frac{dy}{y}$$

Combining relation (31) with inequality (33) and then using Fubini’s theorem to obtain

$$\int_{[0, 1]} \mu([0, y]) \frac{dy}{y} = -\int \log x d \mu(x)$$

By Remark, we have $\mu([0, \infty))=0$. Therefore $\mu([0, y))=-\mu((y, \infty))$ for all $y \geq 0$, and hence

$$\int_1^\infty \mu([0, y]) \frac{dy}{y} = -\int_1^\infty \left( \int_{(y, \infty)} d \mu(x) \right) \frac{dy}{y}$$

$$= -\int_{(1, \infty)} \left( \int_1^y d \mu(x) \right) d \mu(x)$$

$$= -\int_{(1, \infty)} \log x d \mu(x)$$

$$= \int_{(1, \infty)} \log \left( \frac{1}{x} \right) d \mu(x)$$

...(35)

by using Equations (31) and (33).

**Conclusion**

The Calderon’s reproducing formula developed in this paper will be of great importance and useful in inversion problems and wavelet theory on Bessel-Kingman hypergroups.

**Acknowledgment:** The author is very much thankful to the anonymous referee for his valuable suggestions and corrections.
References


Reference # 61J-2009-12-03-01